Bonding curves research questions

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Goal: ask the right questions to move forward the adoption of bonding curves as a financial instrument.

See here for a live version of this doc, open to leaving comments.

Track 1: Are Bonding curves a "Good" Market Mechanism? Price discovery and emergent economic behaviors

Goal: mathematically prove that bonding curves are a well-behaved financial instrument and that they are a suitable alternative to order-book-based trading. Order-book-based trading is the currently prominent method of trading of securities, and bonding curves are an interesting alternative that offers some possible advantages, but only if we prove they are "well-behaved". In general, Bonding curves might be just one of a large family of "automatic market making mechanisms", and possibly an even larger family of market mechanisms. But all such well-behaved mechanisms must enable price discovery.

In particular, here are two properties that everyone assumed bonding curves satisfy, but that no one has proved (AFAIK):

Lemma ("for tiny trades, secondary markets are useless"): Under some assumptions (no latency, anyone is only purchasing tiny amounts,, all players are rational), having secondary markets for bonding curve tokens is useless — any player might as well only trade with the smart contract directly, and be just as profitable.

I added a proof of this Lemma at the bottom of this document.

Conjecture (Price Discovery): Under some assumptions, bonding curves are price-discovering. I.e. if many entities in the market think the "true" value of the underlying asset is X, then trading activity will be at equilibrium when (and only when) the market cap is X.

I think there are many possible ways to prove this, depending on how to exactly formalize the assumption that many players agree on the right price for the underlying. When literally everyone agrees on the price of the underlying, this should be trivial. I'd love to see some proofs of non-trivial cases.
I'd love to find an analogous proof for order-book-based markets, probably in the Mathematical Finance literature, in order to phrase and prove a similar theorem for bonding curves.

Slava gives some related discussion here.

Note: I'm only considering cases where the bonding curve represents a collateralized asset, i.e. where holding tokens gives proportional rights in some underlying asset. For example fractional ownership or a claim to future profits from a revenue stream. I think the analysis works in general when the amount of utility you get from your tokens is proportional to the amount of it that you hold. But it doesn't hold for "binary" utility: where you get something if you hold any amount of the token, and you get no utility if you hold none.
In the rest of this writeup, I always assume that the tokens represent fractional rights in an asset, i.e. if there are X outstanding tokens, then each token holder is entitled to 1/X fraction of the asset. Basically the same as non-voting stock in a company.

Track 2: Are Bonding Curves an "efficient" market mechanism? Bonding curve parameter optimization.

Track 2.1. The Leverage Function

One of the biggest superpowers of financial instruments is *leverage*. When we create a financial instrument, we ask: "given any particular amount of capital, how much value can it represent?". The bigger this ratio, the more "efficient" the instrument. Thus, I suggest that one main way to study bonding curves and other automated market makers is by studying the market cap as a function of the locked capital.
Call this function the Leverage function: **Lev(C)=MC** is the capital that maps the amount of locked capital, **C**, to the market cap, **MC**.
I think this function is way more important to study than the base curve f of the bonding curve, which often produces misleading intuition.

Some examples:

- When the bonding curve is $f(x)=x^n$, the Leverage function is $Lev(C) = (n+1)*C$.
- When the bonding curve is $f(x)=e^x$, the Leverage function is $Lev(C) = C*\ln C$.
- When the bonding curve is $f(x)=e^{(e^x)}$, the Leverage function is $Lev(C) = C*\ln C*\ln\ln C$.
- When the bonding curve is $f(x) = (0 \text{ if } x<t \text{ else } 1)$, then the Leverage function is $Lev(C) = C + t$.
- In general, when the bonding curve is f(x), the Leverage function is the function that maps $\int_{x=0}^{x_0} f(x)$ to $f(x_0) * x_0$.

In comparison, in order-book based markets, the leverage is unbounded. A tiny amount of capital can represent a huge amount of value. In reality, some empirical analyses claim that in practice for large-cap publicly traded companies a typical value for the leverage is $Lev(C) = 40*C$. Note that in the above examples, in order to match this amount of leverage, we'll need to use a fast-growing bonding curve, maybe $f(x)=x^{39}$. Slower-increasing bonding curves might produce leverage that is dramatically lower than that in order book based markets. This means that, hypothetically, companies who want to use bonding curves to trade in equity in their company might find the leverage too small for their purposes.

How big can the leverage get? I believe it can't get too big.

Theorem Sketch (Bonding Curve leverage is bounded): for any bonding curve there is a constant A such that Lev(C)<=A*C*lnC*lnlnC*lnlnlnC*....
(and in particular, Lev(C) <= A * C * lnC * ln^2(lnC).)

The bonding curve that shows this equation is tight is f(x)=e^e^e^e^....^e^x.

Proof Method Sketch. The proof of this theorem basically proceeds by claiming that this function is "the fastest growing function" (for some appropriate definition of the term "fastest growing"). Some work with differential equations is needed.

Example of the Proof Method. To get a bonding curve where Lev(C)=C^2, we'd like to have a base function f with the property that f'(x)=(f(x))^2. (This is a bit tricky to see, but I think it holds.) Solving a differential equation, the only class of functions for which this holds are f(x)=-1/(x+C). These functions are clearly irrelevant for bonding curves, so we find that there are really no base function f that creates such a leverage function, which means that this leverage cannot be obtained via a bonding curve. I think this analysis can be generalized to any Leverage function that grows faster than C*ln(C)*lnln(C)*... .

Track 2.2. Bonding Curve parameter selection

The question: what good quantitative properties do we want a bonding curve based markets to satisfy? This helps us choose a shape for the curve, but also might help us design new variants of bonding curves and other market mechanisms.

Here are my first proposals for what quantities to measure: we want "stability" to be high and "reserve ratio" to be low.

Reserve ratio is the "leverage", meaning the amount of money locked in the contract divided by the market cap. A vehicle with a low reserve ratio allows the market to be highly leveraged, meaning that a small amount of capital enables a large amount of economic activity. This is similar to Burniske's notion of "Fiat Amplifier". Bancor call this ratio the "Reserve Ratio" or the "Connector Weight" (see here or in the Bancor whitepaper).

Stability: means the sensitivity of the asset to sudden shifts in sentiment or to changes in the macroeconomic situation. A stable asset is one that drops slowly and gracefully when some parts of the economy go into panic, and that rises slowly and doesn't create bubbles, when some parts of the economy go into frenzy.

Overall high stability and low reserve ratio are both good things, so we'd like assets that are highly stable and also have low reserve ratio. These are probably have some tradeoff.

Question: Is there a fundamental economic tradeoff between these two things, that can be proved mathematically? Or is it possible to construct an equity-like mechanism that has both high stability and also low reserve ratio?

Now here are my suggested technical definitions for the particular case of bonding curves:

Remark: For the rest of the discussion I assume all bonding curves are fractional and continuous. Otherwise weird discrete jumps happens which I don't care about and screw the analysis.

Suggested Definition: The **reserve ratio** of a bonding curve is the function mapping C to C / Lev(C), where Lev(C) is the leverage function.

Suggested Definition: The **stability** is the function that for any particular amount of locked capital tells us how much market cap we lose if someone withdraws 10% of the locked capital. I.e. it's the function that maps C to Lev(C) / Lev(0.9*C).

Warm-up exercises: Calculate the stability of the curve shapes f(x)=x^k and f(x)=e^(e^(e^x)).

Note that for an order book based market (not bonding curve based), the reserve ratio is arbitrarily close to 0 (good) while the stability can be arbitrarily close to 1 (bad). My suspicious is that bonding curves allow us to jiggle this tradeoff, in order to get more stability at the expense of increasing the reserve ratio.

Question: What bonding curve shape has the lowest possible reserve ratio? And which one balances reserve ratio with stability in the "best" way?

I have some thoughts on this but didn't do all the work. Answering this properly might require some work with differential equations.

A lot of the preceding questions intersect those that can be found in the Bancor paper, except that the Bancor paper addresses them only for the specific Bancor mechanism (i.e. Bonding curves of the form f(x)=x^n), while here we care about any bonding curve shape, and any market mechanism in general.

Observation:

- flat constant bonding curve f(x)=A
  does not give any incentive for market activity at all.

- Super-fast-growing-bonding-curve f(x)=e^e^e^x
  lots of room for market activity, i.e. speculators want to buy super-early. In fact I hypothesize that a super fast growing bonding curve acts exactly like an old fashioned double-sided order-book-based market.
- We can think of bonding curve as something where you could be anywhere between those two extremes — non-market-activity flat and super-markety order-book based marketplace. In the former, the market is like a voucher system, and no bubble can possible grow (and the reserve ratio = 1). In the latter, huge bubbles can potentially form and FOMO and sentiment are fundamental characteristics (and the reserve ratio can be arbitrarily close to 0)

Track 3: Applications and Motivation

Why do we care about bonding curves? I personally care because bonding curves are a heterodox market mechanism, that can replace order book based markets. I also care because to my knowledge, until a few years ago humanity has only known of one market mechanism with the price discovery property: and that is the double-sided order book. And I took it for granted that this is the only mechanism with price discovery. Iif we now find that there are two market mechanisms that both the price discovery property, then there might be many others, which means that financial markets can become much more feature-rich than I previously realized.

More broadly, I care because fintech is hugely important and has world-altering power, and I think we should be very active in exploring new financial primitives/instruments, but to also design them in a very deliberate way. We are in a unique point in time. Blockchain tech is giving us an “research lab” to explore some Fintech primitives that could not appear earlier, because they were difficult to maintain and orchestrate. However, in the last years, the DeFi movement has introduced many examples of new primitives. Furthermore, these primitives seem to me to be revolutionary even when we “re-centralize” them and implement them by a central entity. The question arises: why didn’t Bonding Curves get introduced in contemporary stock markets? No reason at all! They definitely can. So we need to explore these ideas not only for the purposes of blockchain tech, but also for inclusion in the “standard” economy. Matt Levine explained in a really delightful way how crypto is a “sandbox + research lab” for finance.

So, I’d like to research bonding curves and other new financial primitives, aiming to include them as fundamental economic primitives. (Or I’d like someone to tell me I’m drinking the kool-aid and explain why they are actually uninteresting/unexciting/not-novel). To do this, I’d like to make sure to ask the right questions that will develop Bonding Curves and other market mechanisms and allow us to learn how they work.

Appendix: Proof of Lemma “secondary markets are useless”

Lemma (“for tiny trades, secondary markets are useless”):
Under the following assumptions:

1. latency is always 0 and every event happens at a unique point in time.
2. All players are rational and profit-maximizing, inside of the vacuum of the bonding curve, with no outside influences
3. the amount that is bought is always infinitesimally small (call it epsilon).
4. the curve function f is continuous.

It holds that secondary markets for bonding curve tokens is useless — any player might as well only trade with the smart contract directly, and be just as profitable.

Proof of Lemma:

Let's assume that at time t_0 a trade happens in a secondary market, where Alice sells epsilon units to Bob, for price P. We will show that under the assumptions above, at time t_0 Alice can sell to the curve and receive P*(1-delta) money, and Bob can buy from the curve for P*(1+delta) money. (Here, delta is a very small number, which approaches 0 as epsilon tends to 0.) This will show that Alice and Bob might as well sell and buy directly from the curve, and get basically the same price.

To begin the proof, note that assumption #1 means that Alice and Bob are the only players that we need to analyze in this proof. All other players will matter only before or after this trade resolves, and don't matter right now.

Denote by x_0 the number of outstanding tokens in the curve at time t_0. Denote by P_s the revenue from selling epsilon units to the curve at time t_0, and denote by P_b the cost of buying epsilon units from the curve at time t_0. Specifically,
$P_s = \int_{x=x_0-\epsilon}^{x_0} f(x)$
$P_b = \int_{x=x_0}^{x_0+\epsilon} f(x)$

Observation 1: if Alice is rational, then $P \geq P_s$.
Proof: Since latency is 0, Alice can just immediately go sell the epsilon units to the smart contract for a revenue of P_s, instead of selling them to Bob. So the agreed price with Bob must be at least P_s, otherwise the rational Alice would just buy from the curve. QED.

Observation 2: if Bob is rational then $P \leq P_b$.
Proof: analogous to Observation 1.

From Observations 1 and 2, we already get that by Alice selling directly to Bob, both Bob or Alice benefit at most an amount of P_b-P_s over trading directly with the curve. (This only requires assumptions #1 and #2 and might be of independent interest for analyzing bonding curves in practice.)

Observation 3: Denote delta = (P_b - P_s) / P_s. If f is continuous, then lim_{eps -> 0} delta = 0.
Proof: This just follows from the definitions of P_b and P_s and from undergrad calculus. Specifically:
delta = (P_b - P_s) / P_s = P_b / P_s - 1.
Now, P_b and P_s are both functions in eps, and we want to calculate lim_{eps->0} (P_b(eps) / P_s(eps)). Both P_b and P_s separately tend to 0 when eps->0, so we can use L'hopital's rule, and get that
lim_{eps->0} (P_b'(eps) / P_s'(eps)).

Let's derive P_b using its definition:
P_s(eps) = integral_{x=x_0-epsilon}^{x_0} f(x), so P_s'(eps) = f(x_0 - eps)
and similarly P_b'(eps) = f(x_0 + eps).
Thus, lim_{eps->0} (P_b'(eps) / P_s'(eps)) = lim_{eps -> 0} (f(x_0 + eps) / f(x_0 - eps)). Since f is continuous, this is actually equal to f(x_0)/f(x_0) = 1.
Overall ,we get that lim_{eps->0} delta = 0, QED.

Combining Observations (i), (ii) and (iii) we see that P_s <= P <= P_b <= P_s * (1 + delta), and when eps->0, we get that delta -> 0, so all of these numbers are basically equal, and thus Alice and Bob might as well trade directly with the curve. QED.